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Original article

# Arakawa–Lamb Scheme in Application to Stratified Incompressible Fluid in the Absence of Friction

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*Abstract*
*Purpose.* The study is aimed at generalizing the Arakawa–Lamb scheme for discrete equations of the horizontal components of three-dimensional absolute vorticity of an ideal fluid and analyzing its features.
*Methods and Results.* To derive the finite-difference three-dimensional equations of absolute vorticity, a grid containing more unknowns than equations is applied, that permits obtaining the discrete motion equations which, in their turn, yield the equation for absolute vorticity. The resulting expression is presented in the form of three terms reflecting different features of the discrete equations. The first term provides the fulfillment of the energy conservation law for discrete statement, the second term represents the presence of two quadratic invariants for a divergence-free flow, the addition of the third term results in the Arakawa–Lamb scheme under the shallow water approximation. It follows from the presented expression that the second and third terms, which have no analogues in the continuous statement, can be interpreted as a zero approximation with the second order of accuracy. Thus, selection of these expressions makes it possible to construct the schemes with the required features of the conservation laws.
*Conclusions.* The presented form of the discrete equation for three-dimensional absolute vorticity enables the construction of schemes with the desired features. The difference equations for the horizontal components of absolute vorticity are derived, which possess two quadratic invariants.

**Keywords:** Arakawa-Lamb scheme, discrete equations of model, sea dynamics, kinetic energy, absolute vortex, quadratic invariants

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## Introduction

One of the fundamental results in the study of partial differential equations is Noether's theorem [1], which establishes a one-to-one correspondence between the properties of solutions to such systems and the conservation laws they possess. A clear example of its application to shallow water equations is the energy and potential enstrophy preserving scheme (conservation laws), which ensures the constancy of the mean wavenumber weighted by energy (a solution property).

For the two-dimensional dynamics finite-difference system of equations, work [2] derived schemes that preserve energy and the square of vorticity for non-divergent motion. For the shallow water approximation, work [3] presents a discrete system of equations with two quadratic invariants: energy and potential enstrophy. As a consequence of this property, in accordance with the differential formulation, energy transfer towards small scales is prohibited. In [4], the Nambu bracket was discretized while preserving the antisymmetry property. This made it possible to generalize the Arakawa–Lamb scheme and derive a fully discrete (in time and space) finite-difference scheme possessing two quadratic invariants: energy and potential enstrophy. Building on this, we derived explicit finite-difference shallow water equations that conserve mass, circulation, energy and potential enstrophy on both a regular square grid and an unstructured triangular grid. The latter includes a regular hexagonal grid as a special case.

In [5], the classical Arakawa–Lamb scheme, which was originally formulated for orthogonal square grids, is extended to arbitrary non-orthogonal polygonal grids. The scheme obtained in [4] is also generalized to arbitrary orthogonal spherical polygonal grids in such a way as to ensure the conservation of energy and potential enstrophy. For the shallow water equations in the case of generalized curvilinear coordinates, work [6] derives an energy- and potential enstrophy-preserving finite-difference scheme based on tensor analysis. The paper demonstrates that exact conservation of discrete energy and potential enstrophy prevents distortion of the forward and inverse energy cascades in quasi-two-dimensional turbulent flow, thereby enhancing the stability of the scheme.

For shallow water equations that incorporate the full Coriolis force and bottom topography, work [7] presents a scheme conserving energy and potential enstrophy. The authors observe that preserving discrete energy and potential enstrophy prevents distortion of the forward and inverse energy cascades in quasi-two-dimensional turbulence, thereby improving the stability of the scheme.

This paper is a continuation of studies [8, 9] and presents a specific rewriting of the Arakawa–Lamb scheme. This allows the terms responsible for different conservation properties to be explicitly isolated in the equation for absolute velocity vorticity, and the scheme to be generalized for discrete equations of the horizontal components of velocity vorticity.

## Discrete equations of motion

Let us consider the differential equations of an incompressible fluid in a potential force field, assuming the absence of viscosity and external sources. In the Boussinesq approximation, in a Cartesian coordinate system and for a domain $\Omega$ with boundary $\partial\Omega$, the velocity of motion satisfies the following system of equations in in the Gromeka–Lamb form:

$$\frac{\partial \vec{\mathrm{U}}}{\partial t} + \vec{\xi} x \vec{\mathrm{U}} = -\frac{1}{\rho_0}\nabla(P+E) \ + \vec{g}\,\frac{\rho}{\rho_0}, \tag{1}$$

$$\nabla\vec{\mathrm{U}} = 0 \ . \tag{2}$$

The following notations are introduced: $\vec{\mathrm{U}} = (u, v, w)$ – components of the flow velocity vector along the ($x$, $y$, $z$) axes, directed eastward, northward, and vertically downward, respectively; $\vec{g}$ = (0, 0, $g$) – free fall acceleration ; ($P$, $\rho$) – pressure and density of seawater; $\rho_0$ = 1 g/cm$^3$ ((henceforth, pressure and density are assumed to be normalized by $\rho_0$); $\vec{f}$ = $(0, 0, f^z)$ – Coriolis parameter, where $f^z$ = $2\omega\sin\varphi$; $\omega$ – angular velocity of Earth's rotation; $\varphi$ – latitude.

In equation (1), the absolute vorticity and kinetic energy of motion are introduced:

$$\vec{\xi} = \mathrm{rot}\vec{U} + \vec{f}, \quad \xi^x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \xi^y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \xi^z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f^z, \tag{3}$$

$$E = \rho_0 \frac{u^2 + v^2 + w^2}{2}. \tag{4}$$

In terms of tensor analysis,

$$\xi^\alpha = \varepsilon^{\alpha\beta\gamma}\partial_\beta \upsilon^\gamma + f^\alpha, \text{ where } (\upsilon^x, \upsilon^y, \upsilon^z) = (u, v, w).$$

Here and henceforth, $\alpha, \beta, \gamma$ can only take distinct values of $x, y, z$ simultaneously; $\varepsilon^{\alpha\beta\gamma}$ is the Levi-Civita tensor and for each fixed α, summation is performed over β and γ.

$$\text{At } z = 0 \ \ w = -\varsigma_t, \text{ at } z = H(x, y) \ \ w = 0. \tag{5}$$

No-penetration conditions are imposed on lateral walls:
for meridional boundaries $u = 0$, for zonal boundary segments $v = 0$. (6)

Initial conditions:
at $t = t_0$ $u = u^0$, $v = v^0$, $w = w^0$.

The equation for absolute vorticity takes the following form:

$$\frac{\partial \vec{\xi}}{\partial t} + \nabla \times (\vec{\xi} \times \vec{\mathrm{U}}) = \nabla \times (\vec{g}\rho). \tag{7}$$

We approximate the uneven-bottom basin using boxes whose centers correspond to the integer values of the indices $i, j, k$ ($i = i_1, \ldots, i_N$, $j = j_1, \ldots, j_M$, $k = 1, \ldots, K_{i,j}$), while their faces correspond to the values $i+1/2$, $j+1/2$, $k+1/2$. The horizontal box dimensions ($h_x, h_y$) are constant, while the vertical approximation uses non-uniform spacing ($h_z^k = z_{k+1/2} - z_{k-1/2}$, $h_z^{k+1/2} = z_{k+1} - z_k$).

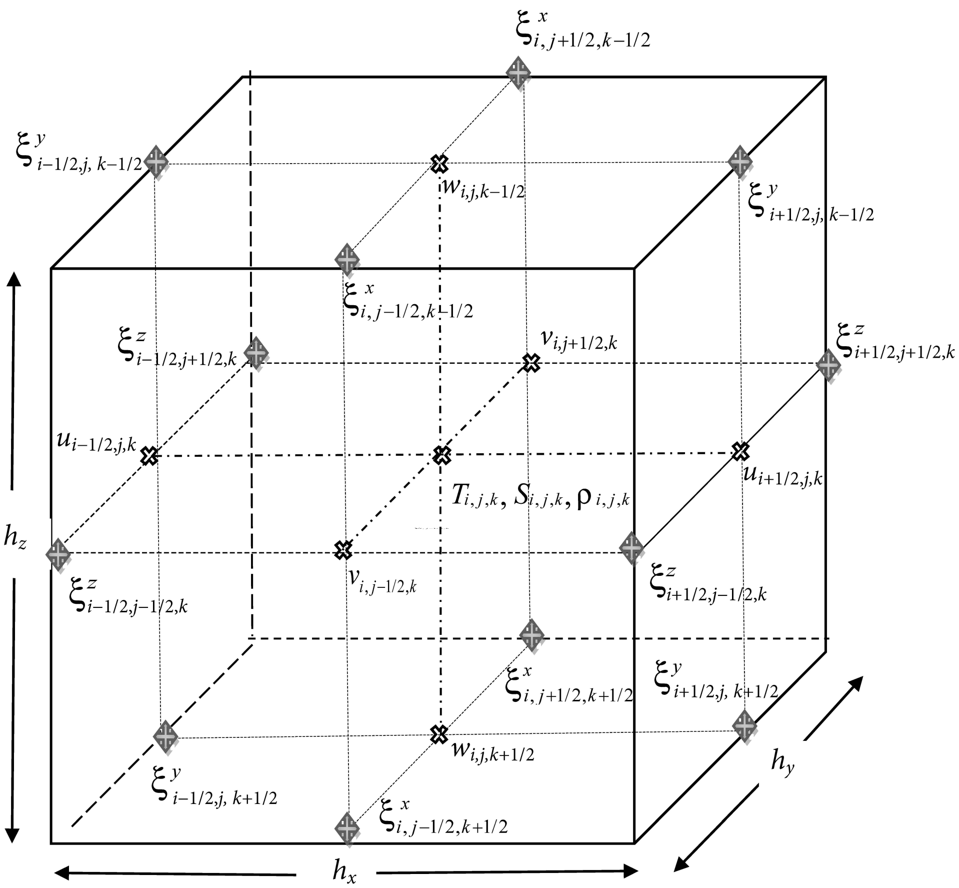
Finite-difference operators are expressed as (similarly for $j$, $k$):

$$\overline{\phi}^x_{i,j,k} = \frac{\phi_{i+1/2,j,k} + \phi_{i-1/2,j,k}}{2}, \quad \delta_x \phi_{i,j,k} = \frac{\phi_{i+1/2,j,k} - \phi_{i-1/2,j,k}}{h_x},$$

$$\nabla^2_{x,y}\phi_{i,j,k} = \delta^2_x \phi_{i,j,k} + \delta^2_y \phi_{i,j,k}, \tag{8}$$

$$\{\phi\}^{\Omega_k} = \frac{1}{\Omega_k}\sum_{i,j}\phi_{i,j,k}h_x h_y, \quad \{\phi\}^V = \frac{1}{V}\sum_{i,j}\sum_{k=1}^{K_{i,j}}\phi_{i,j,k}h_z^k h_x h_y, \quad V = \sum_{i,j}\sum_{k=1}^{K_{i,j}} h_z^k h_x h_y.$$

Horizontal velocity components are calculated at $z_k$ horizons, while vertical velocity is computed at $z_{k+1/2}$ horizons – vertical velocity, $\Omega_k$ denotes the surface area at $k$ horizon. The distribution of the variables is shown in Fig. 1.




**F i g. 1** Distribution of variables in the box (*i, j, k*) and on its edges represents the absolute vorticity components $\xi^x, \xi^y, \xi^z$

The differential-difference equations of motion (differential in time) [3, 9] are written below:

$$\frac{du_{i+1/2,j,k}}{dt} - \left[v, \xi^z\right]_{i+1/2,j,k} + \left[w, \xi^y\right]_{i+1/2,j,k} = -\delta_x (E_{i+1/2,j/k} + P_{i+1/2,j,k}), \tag{9}$$

$$\frac{dv_{i,j+1/2,k}}{dt} + \left[u, \xi^z\right]_{i,j+1/2,k} - \left[w, \xi^x\right]_{i,j+1/2,k} = -\delta_y (E_{i,j+1/2,k} + P_{i,j+1/2,k}), \tag{10}$$

$$\frac{dw_{i,j,k+1/2}}{dt} - \left[u, \xi^y\right]_{i,j,k+1/2} + \left[v, \xi^x\right]_{i,j,k+1/2} = -\delta_z (E_{i,j,k+1/2} + P_{i,j,k+1/2}) + \\ + g\rho_{i,j,k+1/2}. \tag{11}$$

Notations have been introduced for discrete analogs of nonlinear terms:

$$\left[v, \xi^z\right]_{i+1/2,j,k}, \left[w, \xi^y\right]_{i+1/2,j,k}, \left[u, \xi^z\right]_{i,j+1/2,k}, \left[w, \xi^x\right]_{i,j+1/2,k},$$

$\left[u,\xi^{y}\right]_{i,j,k+1/2}, \left[v,\xi^{x}\right]_{i,j,k+1/2}$, whose specific form will be presented later.

In accordance with notations (3), (4), (8), the components of velocity vorticity (see Figure) and kinetic energy take the following form:

$$\xi^{x}_{i,j+1/2,k+1/2} = \delta_y(w_{i,j+1/2,k+1/2}) - \delta_z(v_{i,j+1/2,k+1/2}),$$

$$\xi^{y}_{i+1/2,j,k+1/2} = \delta_z(u_{i+1/2,j,k+1/2}) - \delta_x(w_{i+1/2,j,k+1/2}), \tag{12}$$

$$\xi^{z}_{i+1/2,j+1/2,k} = \delta_x(v_{i+1/2,j+1/2,k}) - \delta_y(u_{i+1/2,j+1/2,k}) + f^{z}_{j+1/2},$$

$$E_{i,j,k} = \frac{\overline{u^2_{i,j,k}}^{x} + \overline{v^2_{i,j,k}}^{y} + \overline{w^2_{i,j,k}}^{z}}{2}.$$

From approximation (5) it follows that at points $i+1/2,\ j+1/2,\ k+1/2$ the following holds:

$$\delta_x\xi^{x}_{i,j+1/2,k+1/2} + \delta_y\xi^{y}_{i+1/2,j,k+1/2} + \delta_z\xi^{z}_{i+1/2,j+1/2,k} = 0. \tag{13}$$

Let us consider motion in the ($x$, $y$) plane. Unlike in the classical work [3], we write the nonlinear term $\left[v,\xi^{z}\right]_{i+1/2,j,k}$ in the first equation and $\left[u,\xi^{z}\right]_{i,j+1/2,k}$ – in the second equation in the following form:

$$\begin{gathered}
-[\xi,v]_{i+1/2,j,k} = -\underbrace{\overline{\overline{v_{i+1/2,j}}^{y}\;\overline{\xi^{z}_{i+1/2,j,k}}^{xy}}^{x}}_{\text{I}} - \underbrace{\frac{h_x^2h_y^2}{48}\delta_x\left[(\delta_y v_{i+1/2,j,k})(\delta_x\delta_y\xi^{z}_{i+1/2,j,k})\right]}_{\text{II}} + \\
+\underbrace{\frac{h_xh_y}{6}\{[\delta_x(\overline{u_{i+1/2,j,k}}^{x}\delta_y\overline{\xi^{z}_{i+1/2,j,k}}^{x})] - \frac{1}{2}[u_{i+1/2,j,k}\delta_x\delta_y\overline{\xi^{z}_{i+1/2,j,k}}^{x}]\}}_{\text{III}},
\end{gathered} \tag{14}$$

$$\begin{gathered}
[\xi,u]_{i,j+1/2,k} = \underbrace{\overline{\overline{u_{i,j+1/2,k}}^{x}\;\overline{\xi^{z}_{i,j+1/2,k}}^{xy}}^{y}}_{\text{I}} + \underbrace{\frac{h_x^2h_y^2}{48}\delta_y\left[(\delta_x u_{i,j+1/2,k})(\delta_x\delta_y\xi^{z}_{i,j+1/2,k})\right]}_{\text{II}} - \\
-\underbrace{\frac{h_xh_y}{6}\{[\delta_y(\overline{v_{i,j+1/2,k}}^{y}\delta_x\overline{\xi^{z}_{i,j+1/2,k}}^{y})] + \frac{1}{2}[v_{i,j+1/2,k}\delta_x\delta_y\overline{\xi^{z}_{i,j+1/2,k}}^{y}]\}}_{\text{III}}.
\end{gathered} \tag{15}$$

After standard transformations taking into account equality (13), we proceed to discrete equations of nondivergent flow in the ($x$, $y$) plane and obtain the equation for velocity vorticity. This equation possesses conservation laws for energy, vorticity and enstrophy (squared vorticity), all of which must be maintained in the finite-difference formulation. The significance of expressing nonlinear terms in equations (9) and (10) as expressions (14) and (15), respectively, is as follows. The finite-difference term marked with the numeral I corresponds to the differential analogue of horizontal advection. In the finite-difference equation for velocity vorticity, it ensures conservation of discrete energy; however, vorticity and enstrophy are not invariants. The term marked with the numeral II is of fourth-order smallness and therefore does not change the order of the finite-difference scheme. However, its presence in expressions (14) and (15) ensures that the discrete equation for velocity vorticity in non-divergent flow satisfies the conservation laws for vorticity, energy and enstrophy [2].

The third term of second-order smallness is fundamentally different from the other two. While the first two can be interpreted as approximations of $v\xi^z$, the third term does not formally correspond to any component. It emerges as a consequence of conservativity being required in the discrete vorticity equation, which, along with the first term, ensures the conservation of energy and potential enstrophy in the shallow water model (divergent motion in the (*x*, *y*) plane) [3].

Based on these considerations, the equations of motion in the shallow water approximation can be expressed as follows:

$$\frac{du_{i+1/2,j}}{dt} - \overline{\overline{v_{i+1/2,j}}^{y}\,\overline{\xi^z_{i+1/2,j}}^{xy}}^{x} = -\delta_x(\eta_{i+1/2,j} + E_{i+1/2,j}) + \Phi^x_{i+1/2,j}, \tag{16}$$

$$\frac{dv_{i,j+1/2}}{dt} + \overline{\overline{u_{i,j+1/2}}^{x}\,\overline{\xi^z_{i,j+1/2}}^{xy}}^{y} = -\delta_y(\eta_{i,j+1/2} + E_{i,j+1/2}) + \Phi^y_{i,j+1/2}, \tag{17}$$

where $\eta_{i,j}$ is the elevation of the free surface. The $\Phi^x_{i+1/2,j}, \Phi^y_{i,j+1/2}$ form is obvious in equations (16) and (17).

Two conclusions can be drawn from the above. Firstly, constructing finite-difference schemes with the required conservation properties may involve selecting suitable expressions of $\Phi^x, \Phi^y$ type, which do not change the order of the problem and can be interpreted as approximating zero with the corresponding order. Secondly, equations (16) and (17) can be expressed using tensor analysis:

$$\frac{d\upsilon^\alpha_{n^\alpha}}{dt} - \varepsilon^{\alpha\beta\gamma}\overline{\overline{\upsilon^\beta_{n^\alpha}}^{\beta}\,\overline{\xi^\gamma_{n^\alpha}}^{\alpha\beta}}^{\alpha} = -\delta^\alpha(h_{n^\alpha} + E_{n^\alpha}) + \Phi^\alpha_{n^\alpha}, \tag{18}$$

where α, β, γ can only take distinct values of *x*, *y*, *z* simultaneously.

Expressions for $\Phi^\alpha_{n^\alpha}$ will be provided later.

The following notations are introduced: $(\upsilon^x_{n^x}, \upsilon^y_{n^y}, \upsilon^z_{n^z}) = (u_{i+1/2,j,k}, v_{i,j+1/2,k}, w_{i+1/2,j+1/2,k})$,

where $n^x, n^y, n^z$ correspond to the points (*i*+1/2, *j*, *k*), (*i*, *j*+1/2, *k*), (*i*, *j*, *k*+1/2).

Setting α = *x* in equation (18) yields equation (16), while setting α = *y* yields equation (17). In other words, swapping *α* and *β* gives us system (16), (17).

Taking the introduced notations into account, we can write equations (9)–(11) as a single equation:

$$\frac{d\upsilon^\alpha_{n^\alpha}}{dt} - \varepsilon^{\alpha\beta\gamma}\overline{\overline{\upsilon^\beta_{n^\alpha}}^{\beta}\,\overline{\xi^\gamma_{n^\alpha}}^{\alpha\beta}}^{\alpha} - \varepsilon^{\alpha\gamma\beta}\overline{\overline{\upsilon^\gamma_{n^\alpha}}^{\gamma}\,\overline{\xi^\beta_{n^\alpha}}^{\alpha\gamma}}^{\alpha} = -\delta_\alpha(E_{n^\alpha} + P_{n^\alpha}) + \vec{g}\rho_{n^\alpha} + \Phi^\alpha_{n^\alpha}. \tag{19}$$

If we set $\alpha = x, \beta = y, \gamma = z$ in equation (19), we obtain:

$$\frac{du_{i+1/2,j,k}}{dt} - \overline{\overline{v_{i+1/2,j,k}}^{y}\,\overline{\xi^z_{i+1/2,j,k}}^{xy}}^{x} + \overline{\overline{w_{i+1/2,j,k}}^{z}\,\overline{\xi^y_{i+1/2,j,k}}^{xz}}^{x} =$$

$$= -\delta_x (E_{i+1/2,j,k} + P_{i+1/2,j,k}) + \Phi^x_{i+1/2,j,k}, \quad (20)$$

at $\alpha = y, \beta = x, \gamma = z$

$$\frac{dv_{i,j+1/2,k}}{dt} + \overline{\overline{u_{i,j+1/2,k}}^{x}\,\overline{\xi^z_{i,j+1/2,k}}^{xy}}^{y} - \overline{\overline{w_{i,j+1/2,k}}^{z}\,\overline{\xi^x_{i+1/2,j,k}}^{yz}}^{y} =$$

$$= -\delta_y (E_{i,j+1/2,k} + P_{i,j+1/2,k}) + \Phi^y_{i,j+1/2,k}, \quad (21)$$

at $\alpha = z, \beta = y, \gamma = x$

$$\frac{dw_{i,j,k+1/2}}{dt} - \overline{\overline{u_{i,j,k+1/2}}^{x}\,\overline{\xi^y_{i,j,k+1/2}}^{xz}}^{z} + \overline{\overline{v_{i,j,k+1/2}}^{y}\,\overline{\xi^x_{i,j,k+1/2}}^{yz}}^{z} =$$

$$= -\delta_z (E_{i,j,k+1/2} + P_{i,j,k+1/2}) + g\rho_{i,j,k+1/2} + \Phi^z_{i,j,k+1/2}. \quad (22)$$

Let us examine the last term in equation (19). We represent it in the form:

$$\Phi^\alpha_{n^\alpha} = \Phi^1_{n^\alpha} + \Phi^2_{n^\alpha} + \Phi^3_{n^\alpha}. \quad (23)$$

Then, in accordance with expressions (14) and (15), the terms in equality (23) can be written as follows:

$$\Phi^1_{n^\alpha} = \varepsilon^{\alpha\beta\gamma} \frac{h^2_\alpha h^2_\beta}{48} \delta_\alpha \left[ (\delta_\beta \upsilon^\beta_{n^\alpha})(\delta_\alpha \delta_\beta \xi^\gamma_{n^\alpha}) \right],$$

$$\Phi^2_{n^\alpha} = -\varepsilon^{\alpha\beta\gamma} \frac{h_\alpha h_\beta}{6} [\delta_\alpha (\overline{\upsilon^\alpha_{n^\alpha}}^{\alpha} \delta_\beta \overline{\xi^\gamma_{n^\alpha}}^{\alpha})], \quad (24)$$

$$\Phi^3_n = \varepsilon^{\alpha\beta\gamma} \frac{h_\alpha h_\beta}{12} [\upsilon^\alpha_{n^\alpha} \delta_\alpha \delta_\beta \overline{\xi^\gamma_{n^\alpha}}^{\alpha}].$$

Note that, from this point onwards, summation is performed for a fixed α over two permutations: β, γ and γ, β. Therefore, taking expression (24) into account, equations (21) and (22) can be written as a single equation:

$$\frac{d\upsilon^\alpha_{n^\alpha}}{dt} \underbrace{- \varepsilon^{\alpha\beta\gamma} \overline{\overline{\upsilon^\beta_{n^\alpha}}^{\beta}\,\overline{\xi^\gamma_{n^\alpha}}^{\alpha\beta}}^{\alpha}}_{\text{I}} + \delta_\alpha (E_{n^\alpha} + P_{n^\alpha}) - \vec{g}\rho_{n^\alpha} = \underbrace{\varepsilon^{\alpha\beta\gamma} \frac{h_\alpha h_\beta}{12} \left[ \upsilon^\alpha_{n^\alpha} \delta_\alpha \delta_\beta \overline{\xi^\gamma_{n^\alpha}}^{\alpha} \right]}_{\text{II}} +$$

$$\underbrace{+ \varepsilon^{\alpha\beta\gamma} \frac{h^2_\alpha h^2_\beta}{48} \left( \delta_\alpha \left\langle (\delta_\beta \upsilon^\beta_{n^\alpha})(\delta_\alpha \delta_\beta \xi^\gamma_{n^\alpha}) \right\rangle - 2\delta_\alpha (\overline{\upsilon^\alpha_{n^\alpha}}^{\alpha} \delta_\beta \overline{\xi^\gamma_{n^\alpha}}^{\alpha}) \right)}_{\text{III}}. \quad (25)$$

The discrete equation (25) possesses the following features. Term I is a discrete analogue of the nonlinear term in the equation of motion. In the shallow water approximation, it satisfies the energy conservation law, but does not guarantee the conservation of potential enstrophy (the second quadratic invariant). Adding the term marked as II yields a scheme that provides two discrete invariants: energy and enstrophy (squared vorticity) for non-divergent flow. However, the property of potential enstrophy conservation in the shallow water approximation is still absent. Finally, term III yields the Arakawa–Lamb scheme [3].

Although the right-hand side does not formally correspond to the differential form of equation (1), yet it does not alter the order of the scheme and can be interpreted as a second-order accurate spatial approximation of zero.

In the specified notation, the absolute velocity vorticity (12) can be rewritten as a finite-difference analogue of (3):

$$\xi^{\alpha}_{n^{\alpha}} = \varepsilon^{\alpha\beta\gamma}\delta_{\beta}\upsilon^{\gamma}_{n^{\alpha}} + f^{\alpha}. \tag{26}$$

By performing the appropriate operations, we obtain a differential-difference (time-differential) equation for absolute velocity vorticity (a discrete analogue of equation (7):

$$\frac{d\xi^{\alpha}_{n^{\alpha}}}{dt} + \delta_{\beta}\left[\upsilon^{\beta}_{n^{\beta}}, \xi^{\alpha}_{n^{\alpha}}\right] - \delta_{\gamma}\left[\upsilon^{\alpha}_{n^{\alpha}}, \xi^{\gamma}_{n^{\gamma}}\right] = \vec{g}\vartheta^{\alpha}_{n^{\alpha}} - \frac{\varepsilon^{\alpha\beta\gamma}}{2}\left[\delta_{\beta}(\Phi^{\gamma}_{n^{\gamma}}) - \delta_{\gamma}(\Phi^{\beta}_{n^{\beta}})\right]. \tag{27}$$

**Quasi-static approximation**

Let us consider a special case of motion in the quasi-static approximation, taking into account the boundary conditions (6) and initial conditions. In this case,

$$\xi^{x} = -\frac{\partial v}{\partial z}, \quad \xi^{y} = \frac{\partial u}{\partial z}, \quad \xi^{z} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f^{z}, \quad E = \rho_0 \frac{u^2 + v^2}{2}. \tag{28}$$

The continuity equation retains its original form (2).

Note that we assume $\rho_0 = 1\, g / cm^3$.

In accordance with the C-grid (Fig. 1), the finite-difference analogues of velocity vorticity (26) in the form (28), as well as the kinetic energy, can be written as follows:

$$\xi^{x}_{i,j+1/2,k+1/2} = -\delta_z(\mathrm{v}_{i,j+1/2,k+1/2}), \quad \xi^{y}_{i+1/2,j,k+1/2} = \delta_z(u_{i+1/2,j,k+1/2}), \tag{29}$$

$$\xi^{z}_{i+1/2,j+1/2,k} = \delta_x(v_{i+1/2,j+1/2,k}) - \delta_y(u_{i+1/2,j+1/2,k}) + f^{z}_{j+1/2}.$$

$$E_{i,j,k} = \frac{\overline{u^2_{i,j,k}}^{x} + \overline{\mathrm{v}^2_{i,j,k}}^{y}}{2}.$$

The discrete equation of motion (25), with $\alpha = x$, $\beta = y$, $\gamma = z$ and $\alpha = y$, $\beta = x$, $\gamma = z$, leads to the system of equations (20) and (21), where the components of absolute vorticity take the form (29).

Let us rewrite the obtained equations as follows:

$$\frac{du_{i+1/2,j,k}}{dt} + N^{u}_{i+1/2,j,k}(u,v) + \left(\overline{\overline{w}^{x}_{i+1/2,j,k}\xi^{y}_{i+1/2,j,k}h^{k}_{z}}^{z}(h^{k}_{z})^{-1}\right) = -\delta_x(E_{i+1/2,j/k} + P_{i+1/2,j,k}), \tag{30}$$

$$\frac{dv_{i,j+1/2,k}}{dt} + N^{v}_{i,j+1/2,k}(u,v) - \left(\overline{\overline{w}^{y}_{i,j+1/2,k}\xi^{x}_{i,j+1/2,k}h^{k}_{z}}^{z}(h^{k}_{z})^{-1}\right) = -\delta_y(E_{i,j+1/2,k} + P_{i,j+1/2,k}). \tag{31}$$

To describe vertical advection, we selected a scheme based on a minimal difference stencil to ensure energy conservation in the equations of motion. This is a special case of the scheme presented in equations (21)–(24). The horizontal advective terms in equations (30) and (31) can be verified to have the following form:

$$N^u_{i+1/2,j,k}(u,v) = -(\alpha^1_{i+1,j+1/2,k} v_{i+1,j+1/2,k} + \alpha^2_{i+1,j-1/2,k} v_{i+1,j-1/2,k} + \alpha^3_{i,j+1/2,k} v_{i,j+1/2,k} +$$
$$+\alpha^4_{i,j-1/2,k} v_{i,j-1/2,k}) + \alpha^5_{i+3/2,j,k} u_{i+3/2,j,k} - \alpha^6_{i-1/2,j,k} u_{i-1/2,j/k},$$

$$N^v_{i,j+1/2,k}(u,v) = \beta^1_{i+1/2,j+1,k} u_{i+1/2,j+1,k} + \beta^2_{i-1/2,j+1,k} u_{i-1/2,j+1,K} + \beta^3_{i+1/2,j,k} u_{i+1/2,j,k} +$$
$$+\beta^4_{i-1/2,j,k} u_{i-1/2,j,k}) - \beta^5_{i,j+3/2,k} v_{i,j+3/2,k} + \beta^6_{i,j-1/2,k} v_{i,j-1/2,k}, \qquad (32)$$

where

$$\alpha^1_{i+1,j+1/2,k} = \frac{1}{24}(2\xi^z_{i+3/2,j+1/2,k} + \xi^z_{i+3/2,j-1/2,k} + \xi^z_{i+1/2,j+1/2,k} + 2\xi^z_{i+1/2,j-1/2,k}),$$

$$\alpha^2_{i+1,j-1/2,k} = \frac{1}{24}(\xi^z_{i+3/2,j+1/2,k} + 2\xi^z_{i+3/2,j-1/2,k} + 2\xi^z_{i+1/2,j+1/2,k} + \xi^z_{i+1/2,j-1/2,k}),$$

$$\alpha^3_{i,j+1/2,k} = \frac{1}{24}(\xi^z_{i+1/2,j+1/2,k} + 2\xi^z_{i+1/2,j-1/2,k} + 2\xi^z_{i-1/2,j+1/2,k} + \xi^z_{i-1/2,j-1/2,k}),$$

$$\alpha^4_{i,j-1/2,k} = \frac{1}{24}(2\xi^z_{i+1/2,j+1/2,k} + \xi^z_{i+1/2,j-1/2,k} + \xi^z_{i-1/2,j+1/2,k} + 2\xi^z_{i-1/2,j-1/2,k}),$$

$$\alpha^5_{i+3/2,j,k} = \frac{1}{24}(\xi^z_{i+3/2,j+1/2,k} - \xi^z_{i+1/2,j-1/2,k} + \xi^z_{i+1/2,j+1/2,k} - \xi^z_{i-1/2,j-1/2,k}),$$

$$\alpha^6_{i-1/2,j,k} = \frac{1}{24}(\xi^z_{i+1/2,j+1/2,k} - \xi^z_{i+1/2,j-1/2,k} + \xi^z_{i-1/2,j+1/2,k} - \xi^z_{i-1/2,j-1/2,k}), \qquad (33)$$

$$\beta^1_{i+1/2,j+1,k} = \frac{1}{24}(2\xi^z_{i+1/2,j+3/2,k} + \xi^z_{i+1/2,j+1/2,k} + \xi^z_{i-1/2,j+3/2,k} + 2\xi^z_{i-1/2,j+1/2,k}),$$

$$\beta^2_{i-1/2,j+1,k} = \frac{1}{24}(\xi^z_{i+1/2,j+3/2,k} + 2\xi^z_{i+1/2,j+1/2,k} + 2\xi^z_{i-1/2,j+3/2,k} + \xi^z_{i-1/2,j+1/2,k}),$$

$$\beta^3_{i-1/2,j,k} = \frac{1}{24}(\xi^z_{i+1/2,j+1/2,k} + 2\xi^z_{i+1/2,j-1/2,k} + 2\xi^z_{i-1/2,j+1/2,k} + \xi^z_{i-1/2,j-1/2,k}),$$

$$\beta^4_{i-1/2,j,k} = \frac{1}{24}(2\xi^z_{i+1/2,j+1/2,k} + \xi^z_{i+1/2,j-1/2,k} + \xi^z_{i-1/2,j+1/2,k} + 2\xi^z_{i-1/2,j-1/2,k}),$$

$$\beta^5_{i,j+3/2,k} = \frac{1}{24}(\xi^z_{i+1/2,j+3/2,k} - \xi^z_{i-1/2,j+3/2,k} + \xi^z_{i+1/2,j+1/2,k} - \xi^z_{i-1/2,j+1/2,k}),$$

$$\beta^6_{i,j-1/2,k} = \frac{1}{24}(\xi^z_{i+1/2,j+1/2,k} - \xi^z_{i-1/2,j+1/2,k} + \xi^z_{i+1/2,j-1/2,k} - \xi^z_{i-1/2,j-1/2,k}).$$

The equation for the vertical component of absolute velocity vorticity at the point $i+1/2, j+1/2, k+1/2$ is written as ($\alpha = z$ in the equation (27)):

$$\frac{d\xi^z}{dt} + \delta_x[N^u(u,v)] + \delta_y([N^v(u,v)]) - \delta_x([w,\xi^x]) - \delta_y([w,\xi^y]) = 0. \quad (34)$$

The form of the last two terms in equation (34) is clear.

Approximation (32)–(34) corresponds exactly to the Arakawa–Lamb scheme. Therefore, in the shallow water approximation, equation (34) possesses two quadratic invariants: energy and potential enstrophy. When these two quadratic conservation laws are satisfied, the mean wavenumber remains time-independent. This consequently prevents systematic energy transfer to motions with high wavenumbers, thereby enhancing the stability of the numerical solution.

The formulation obtained in equations (25) and (27) enables us to derive analogous schemes for the other two velocity vorticity components.

## Conclusion

This paper presents the Arakawa–Lamb scheme as comprising three distinct terms that reflect the different properties of the discrete equations. The first term ensures energy conservation in the discrete formulation; the second leads to a scheme with two quadratic invariants for non-divergent flow; and the addition of the third term corresponds to the full Arakawa–Lamb scheme. A crucial feature of this representation is that the second and third terms are not directly analogous to any terms in the system of differential equations. Although they do not affect the order of approximation, they do significantly influence the properties of the scheme. As the grid spacing decreases, these terms tend towards zero and can therefore be interpreted as being equal to zero when expressed as a function $h^2\varphi_{i,j,k}$. Consequently, appropriate finite-difference approximations of the zero right-hand side can be selected to construct difference schemes with various conservation properties. As there is an infinite number of such variants, it is necessary to develop a formalism that can identify schemes with specific characteristics.

The fundamental result is that the presented formulation enables the derivation of difference equations for the horizontal vorticity components, which, like the Arakawa–Lamb scheme, possess two quadratic invariants.

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